ON BISECTION-TYPE ALGORITHMS FOR FE MESH ADAPTIVITY

Sergey Korotov

Bergen University College

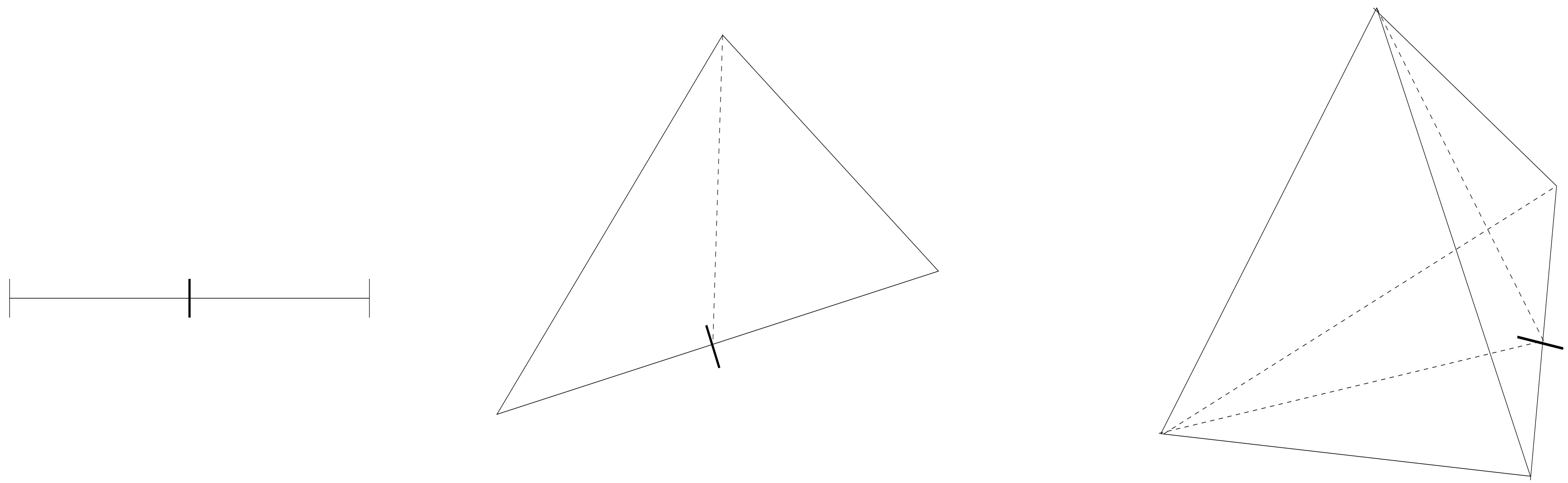
The talk is based on the joint work with my colleagues

A. Hannukainen (Helsinki) & M. Křížek (Prague)

Bisections

We shall only consider simplices in what follows ...

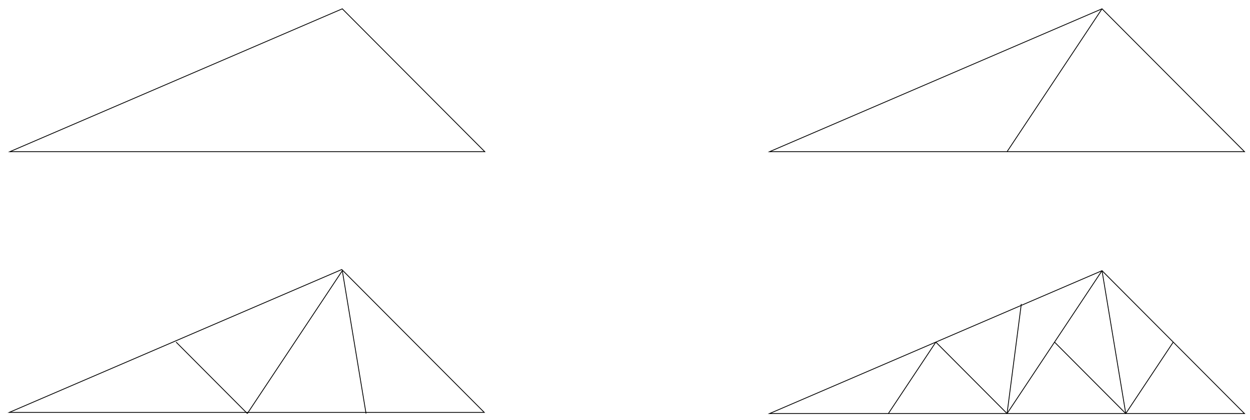
- (Arbitrary) bisection and longest-edge (LE-) bisection algorithms:



- First applications: Find the root $\mathbf{x}_\star$ of equation $\mathbf{F}(\mathbf{x}) = \mathbf{0}$
- Works of 1970–1983: Stynes, Sikorski, Stenger, Kearfott, Adler, ...

Classical Bisections For Triangular Partitions

Main idea: Bisection is at the same time applied to each subtriangle in the current partition, i.e. the number of triangles doubles at each step



`Bisecting to the longest side (LE-bisection) seems to be really attractive (e.g. for finding the roots, etc) as one may thus avoid producing too big and too small angles always undesirable due to various (computational) reasons`

Classical LE–Bisection: Main Results

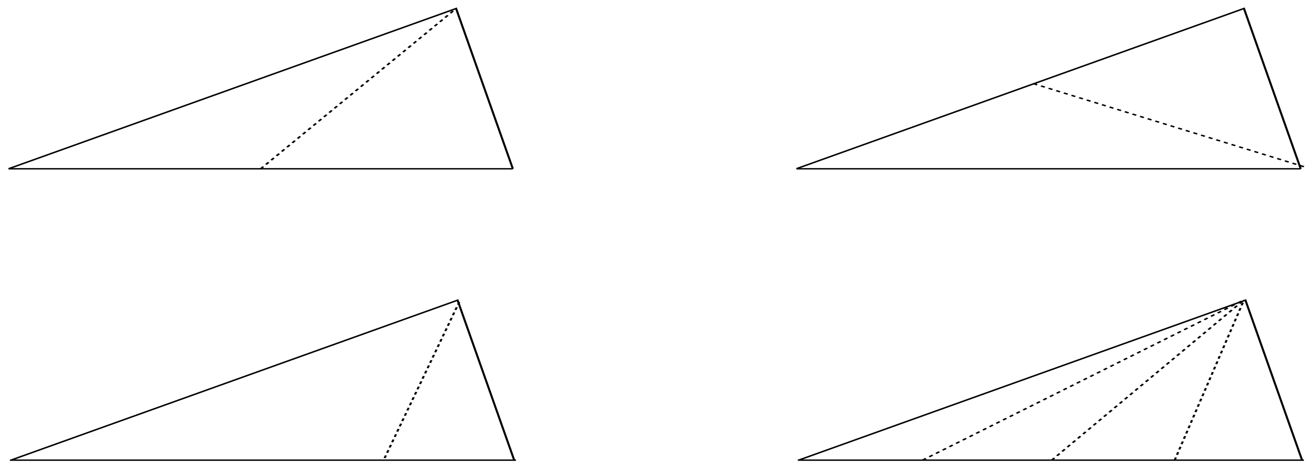
Bisection always only towards the longest edges (as a very precisely defined procedure) delivers certain benefits for a priori analysis:

- Rosenberg & Stenger (1975) - no angle of no triangle tends to zero

- Kearfott (1978) - the largest diameter of all newly generated simplices tends to zero for an arbitrary dimension

- Stynes (1979–80), Adler (1983) - only a finite number of similarity-distinct subtriangles is produced

There are also some bisection-like algorithms which halve not necessarily the longest edges, or not always halving the edges (longest or non-longest ones) in principle, it can even be a trisection, n-section, etc ...



Remark: `For` n`-section algorithms (`$n \geq 3$`) many interesting results have been recently obtained by the group of scientists in Las Palmas - Ángel Plaza, José Suárez et al.`

• We shall mainly concentrate on those features of the bisection-type algorithms which are relevant to FEM context

Remark: `It is worth to mention here that bisection algorithms were advised to use for FEMs already in 1975-79 by Rosenberg, Stenger, and Stynes, however without any concrete details and analysis, probably due to the problem of the so-called hanging nodes (this will be discussed later) ...`

• Next, we shortly remind the main ideas behind the standard FEM

Standard FEM Procedure

PDE model: Find u such that

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega$$

Weak formulation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx \qquad \forall w \in H_0^1(\Omega)$$

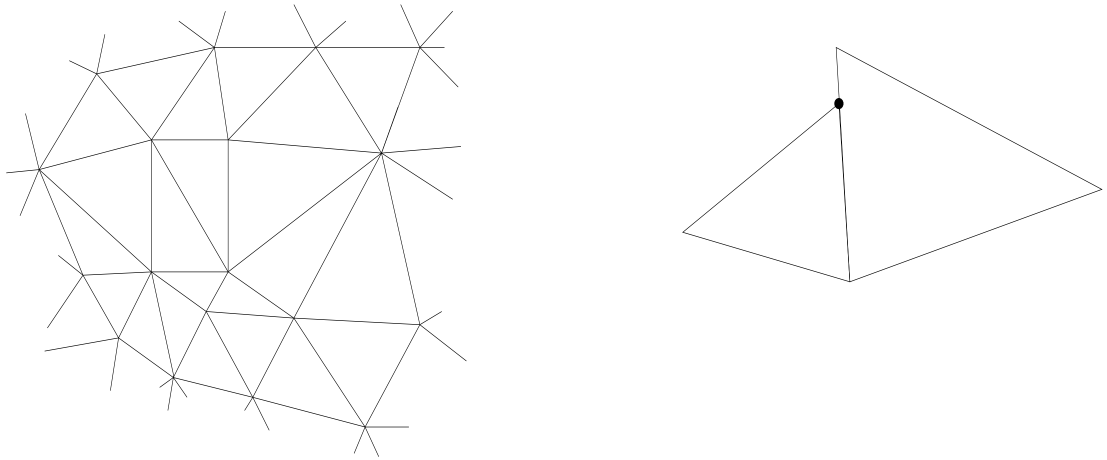
FE scheme: Find $u_h \in V_h \subset H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u_h \cdot \nabla w_h \, dx = \int_\Omega f w_h \, dx \quad \forall w_h \in V_h$$

- Finite-dimensional space V_h is often constructed using certain (e.g. simplicial) mesh $\mathcal{T}_h$ (sometimes called a triangulation) over domain $\overline{\Omega}$, where the parameter h stands for the characteristic size of $\mathcal{T}_h$. It is generally defined as the length of the longest edge in the mesh $\mathcal{T}_h$

On Conformity of FE Meshes

- For standard (classical) FEM the triangulations must be conforming (i.e. without hanging nodes)



Remark: `Conforming triangulations guarantee continuity of (linear) FE approximations` u_h, `i.e.,` $u_h \in H_0^1(\Omega)$

- In what follows we consider only conforming FE meshes

Convergence Analysis for FEM

Principal convergence of computed approximations is the basic requirement for any meaningful numerical scheme !

- Convergence in FE analysis is usually proved under certain mesh regularity assumptions

Definition: An infinite set $\mathcal{F}$ of triangulations of $\overline{\Omega}$ is called a *family of triangulations* if for any $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$

- We shall use the following denotation $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$

The Inscribed Ball Condition

Definition: A family of triangulations $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ is called *regular* if there exists a constant $\kappa > 0$ such that for any triangulation $\mathcal{T}_h \in \mathcal{F}$ and for any triangle $T \in \mathcal{T}_h$ there exists a ball $\mathcal{B}_T \subset T$ of a radius ρ_T such that

$$\frac{\rho_T}{h_T} \geq \kappa,$$

where $h_T = \operatorname{diam} T$

Theorem: The following error estimate for linear FE approximations of our PDE model holds

$$\|u - u_h\|_{1,\Omega} \leq Ch|u|_{2,\Omega}$$

for sufficiently small h, provided $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ is regular and $u \in H^2(\Omega)$

Zlámal's Mimimum Angle Condition

Definition: A family $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ of triangulations is called *regular* if there exists a constant $\alpha_0 > 0$ such that for all triangulations $\mathcal{T}_h \in \mathcal{F}$ and for all triangles $T \in \mathcal{T}_h$ we have

$$\alpha_T \geq \alpha_0 > 0$$

where α_T is the smallest angle in T

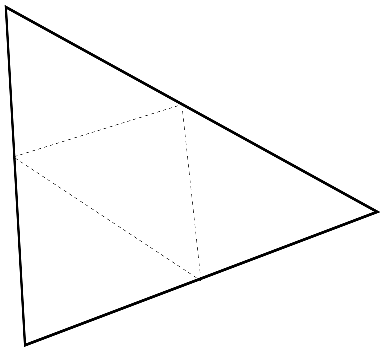
Theorem: The inscribed ball condition and Zlámal's minimum angle condition are equivalent (in two-dimensional case)

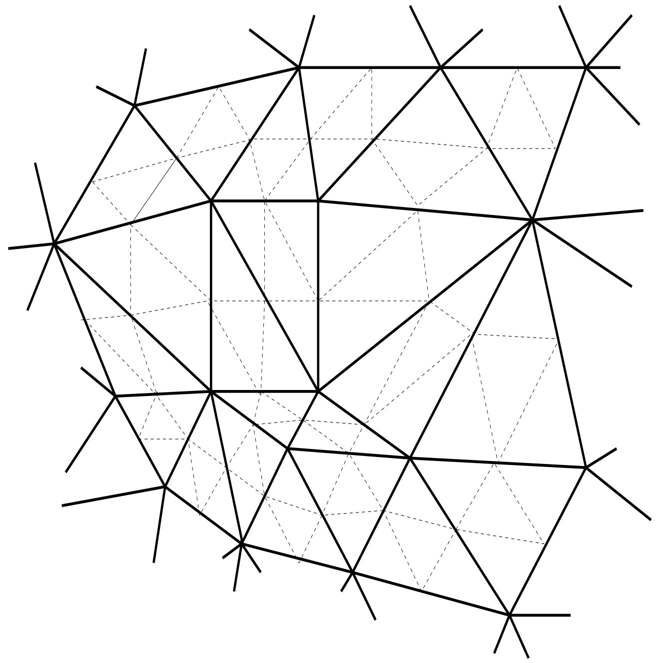
Remark: `There are equivalent analogues of both conditions in higher dimensions`

• Thus, convergence of linear FE approximations in $2d$ strongly depends on geometric features of triangulations: values of the parameter h and value of the minimal angle α_0 !

• In order to provide convergence of linear FE approximations in two-dimensional case one should be able to construct the sequence of triangulations $\mathcal{T}_{h_0}$, $\mathcal{T}_{h_1}$, $\mathcal{T}_{h_2}$, ..., where $h_i \to 0$ monotonically as $i \to \infty$, so that all triangles in all the meshes generated “do not shrink”.

$2d$ Red Refinement

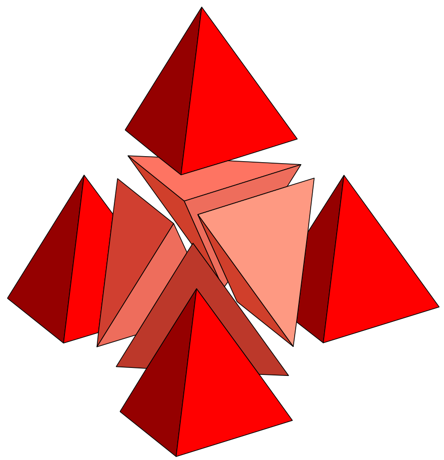




- Triangles do not shrink, the mesh size changes as h, $h/2$, $h/4$, ...

• The above construction is good theoretically but not always suitable in practice, especially for construction of (economic) adaptive FE meshes

• $3d$ analogue of red refinement "works" already essentially differently as no similarity effect exists (only possible in one special case)



• Assosiated elements in the successive refinements differ too much in volume (by factor 4 - in 2D, by factor 8 - in 3D, etc ...), which might be undesirable in some situations, e.g. if one needs finer control over meshes

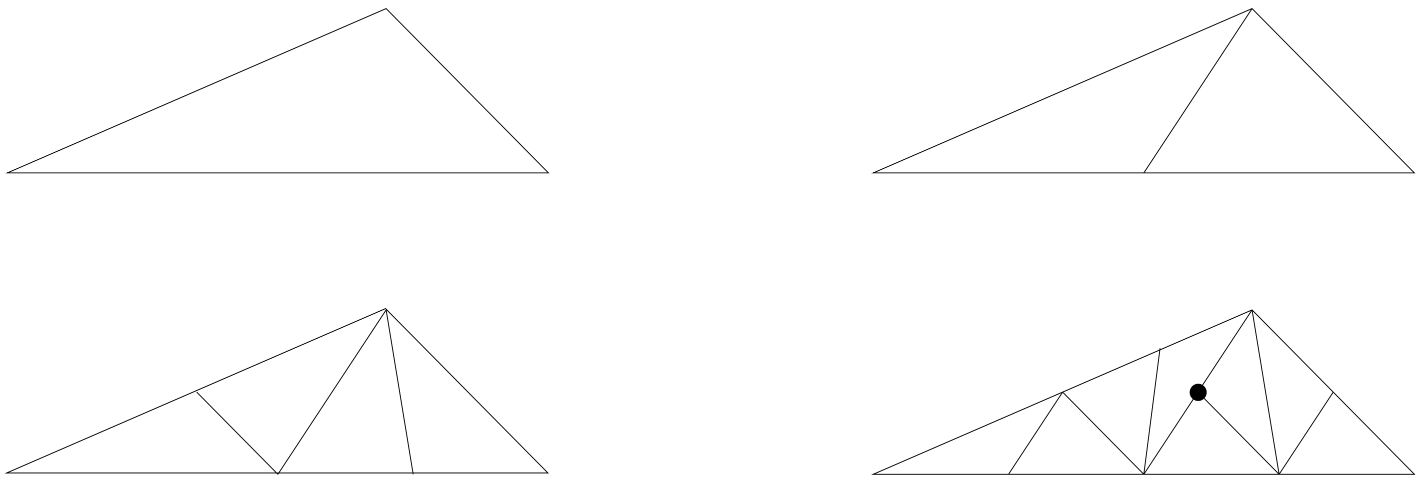
• In many ways, bisection algorithms can be really a good alternative then (more slow division of volume - by factor 2 in all dimensions, simplicity in coding, etc)

• We may also try to apply bisections for some selected triangles only in order to construct adaptive meshes. Bisection may be good in this respect as it produces only 1 new vertex per step, and the red refinement produces 3 (in 2D), and even more (in higher dimensions) ...

Simultaneous Bisections of All Triangles

May Produce Hanging Nodes

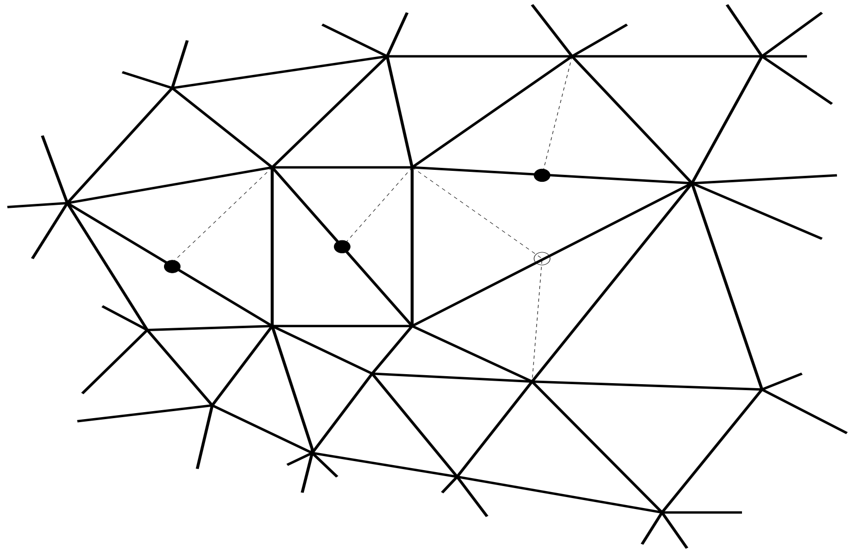
In general, hanging nodes appear during classical bisections



Therefore, the classical bisection algorithms are not always suitable for mesh generation purposes for standard (conforming) FEMs, at least without modification

Bisections of Some Selected Triangles May

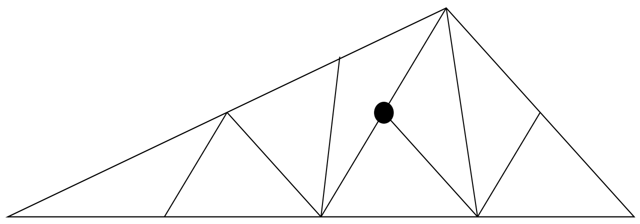
Also Produce Hanging Nodes

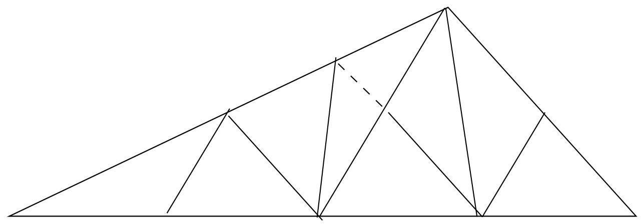


Probably, the problem of hanging nodes was a main reason of absense of a serious analysis of a very high potential of bisection techniques for FEMs till the mid of 80-th !

On Works of M.-C. Rivara

`To force conformity by extra bisections is an obvious idea ?`

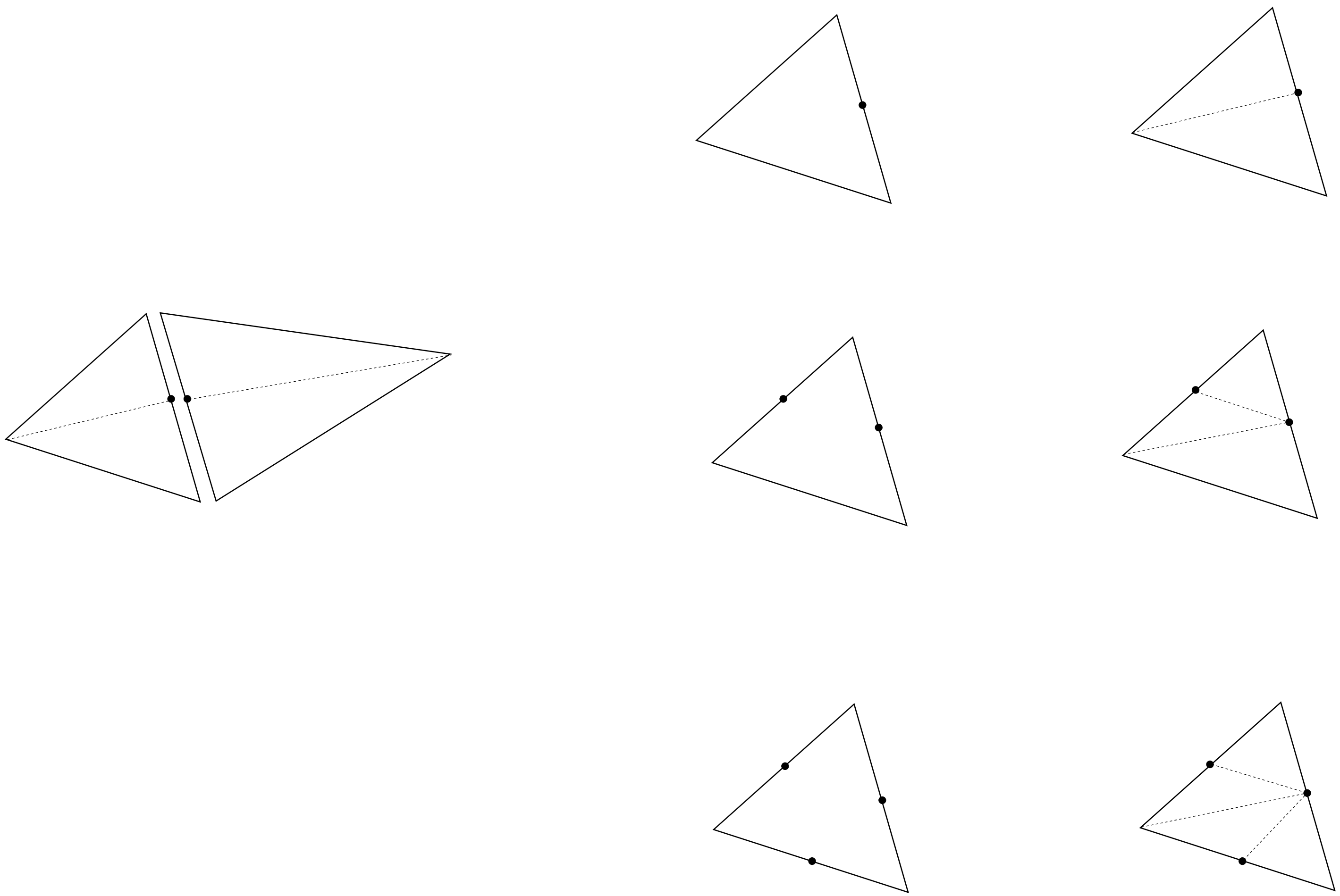




However, this topic started to be seriously analysed only since 1984 in works by M.-C. Rivara, see:

M.-C. Rivara. Algorithms for refining triangular grids suitable for adaptive and multigrid techniques. Internat. J. Numer. Methods Engrg. ***20*** *(1984)*

On Forced Mesh Conformity by Bisections



Properties of Rivara's Algorithms

Several different algorithms (in the above spirit) have been proposed for both, local and global, mesh refinements, see also another her work:

M.-C. Rivara. Selective refinement/derefinement algorithms for sequences of nested triangulations. Internat. J. Numer. Methods Engrg. ***28*** *(1989)*

- Guaranteed conformity after a finite number of post-refinements
- Non-degeneracy (regularity) of meshes: $\alpha \geq \frac{\alpha_0}{2} = const > 0$
- Smoothness: For any adjacent triangles T_1 and T_2, one has

$$\frac{\min(h_{T_1}, h_{T_2})}{\max(h_{T_1}, h_{T_2})} \geq const > 0$$

Remarks on Rivara's Algorithms

The above properties were supported by many numerical tests presented by Rivara and her coauthors. Also in 3d, see:

M.-C. Rivara, C. Levin. A 3D refinement algorithm suitable for adaptive and multigrid techniques. Comm. Appl. Numer. Methods Engrg. ***8*** *(1992)*

• Works of M.-C. Rivara led to a number of other important publications on the usage of bisection algorithms for FE methods

More Results on Bisections for FEMs

• Several another algorithms in the same spirit (with strict proofs, also in higher dimensions) were later developed in papers by Bänsch (1991), Kossaczký (1994), Liu, Joe (1994–1996), Maubach (1995–1996), Traxler (1997), Horst (1997), Arnold, Mukherjee, Pouly (2000), ... Also in a series of various joint papers by Plaza, Carey, Rivara, Falcón, Suárez, Padrón, with several coauthors during last 15 years

• Some of works were oriented to adaptive mesh reconstruction after a posteriori error analysis (after some "selected" elements (i.e. not necessarily all) have been bisected), and therefore include a very non-trivial post-refinement procedure of "conforming mesh closure"

• In general, the algorithms designed so far for post-refinements (to provide conformity) are quite complicated algorithmically

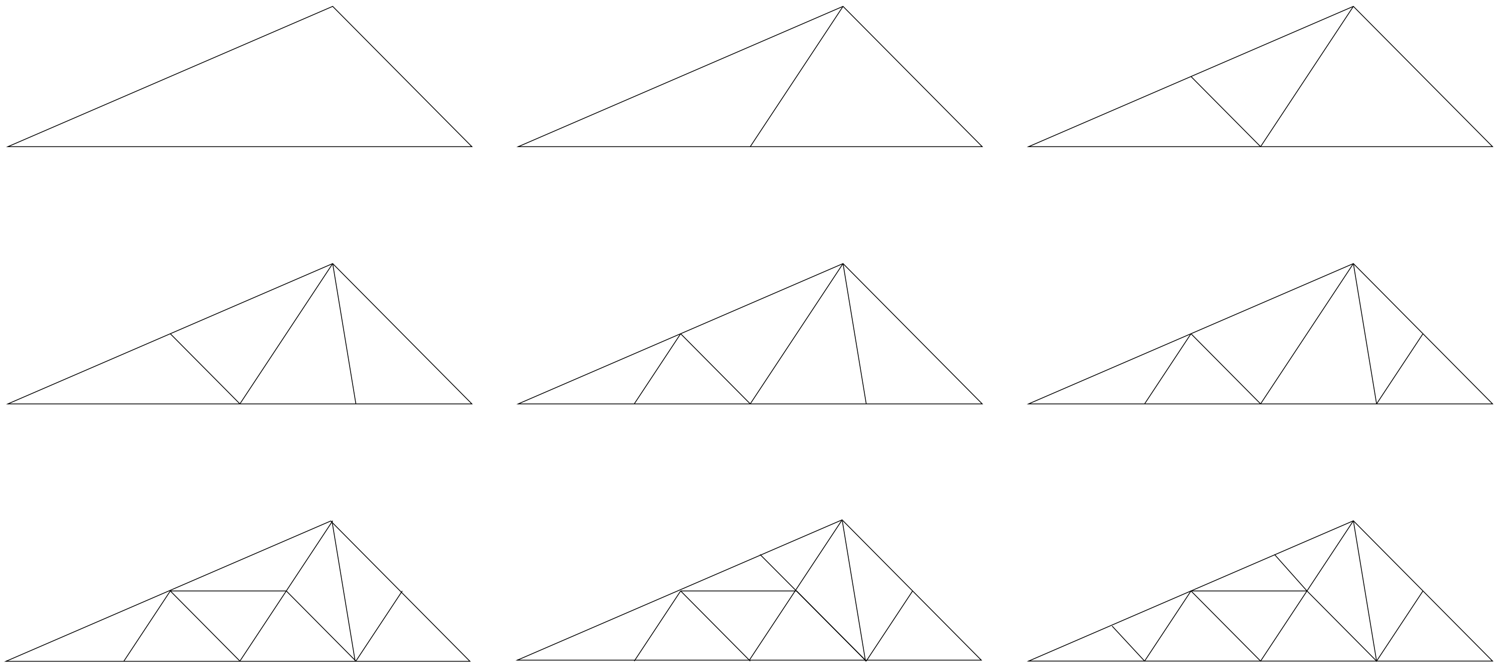
• Some algorithms have serious drawbacks, for example, in order to refine just one element they may refine in some situations all (or almost all) other elements in the whole current mesh ...

`Further, we will present here some recently developed (and seemingly new ?) global and local refinement bisection algorithms not having this type of problems at all and which are suitable in any dimension. We shall also present most useful properties of generated triangulations (all are proved theoretically !), discuss some representative tests, and formulate a few open relevant problems`

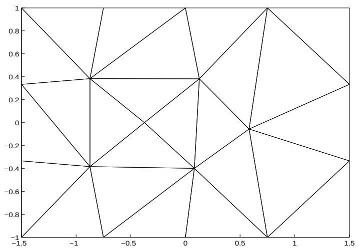
Conforming LE-Bisection (CLEB)

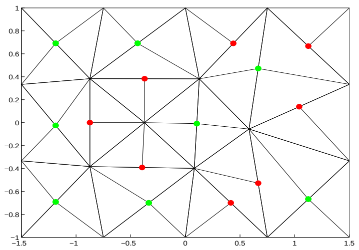
In *S. Korotov, M. Křížek, A. Kropáč. Strong regularity of a family of face-to-face partitions generated by the longest-edge bisection algorithm. Comput. Math. Math. Phys.* ***48*** *(2008)* a new conforming LE-bisection (CLEB) algorithm was proposed

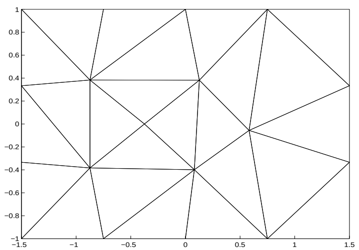
Main idea: to bisect at each step only those (simplicial) elements which surround the longest edge in the whole simplicial partition. CLEB does not produce hanging nodes at all !

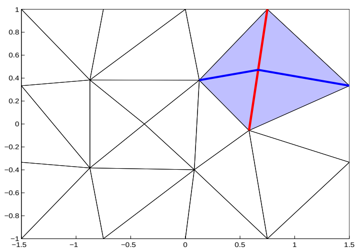


Hanging Nodes Do Not Appear

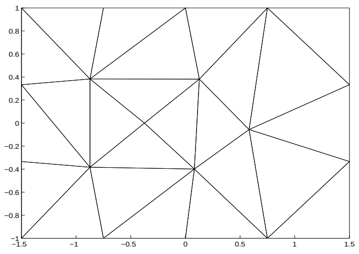


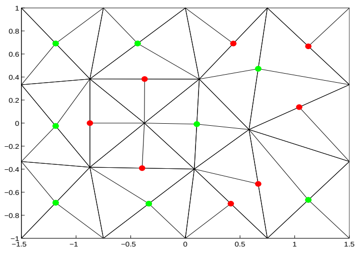


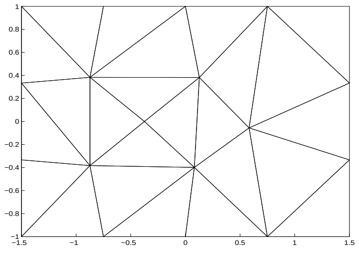


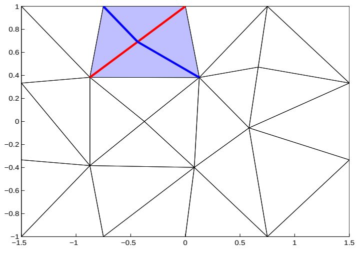


Hanging Nodes Do Not Appear

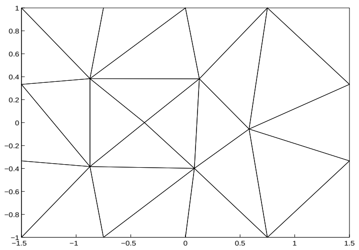


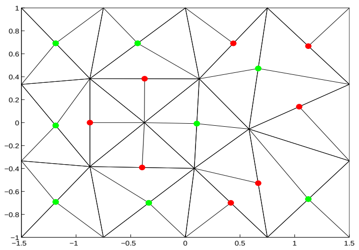


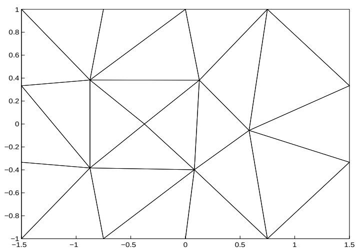


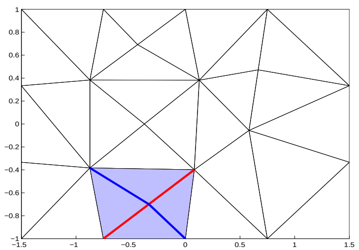


Hanging Nodes Do Not Appear

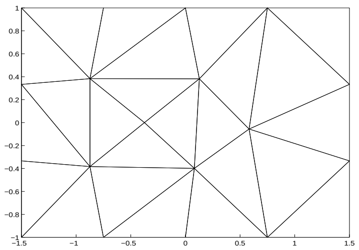


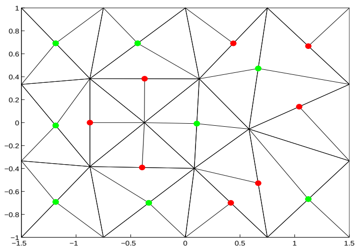


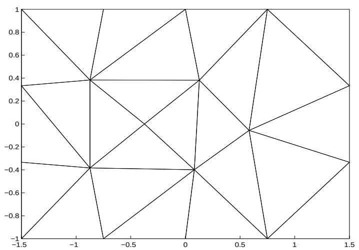


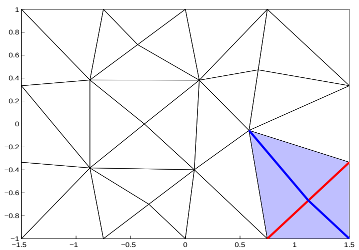


Hanging Nodes Do Not Appear

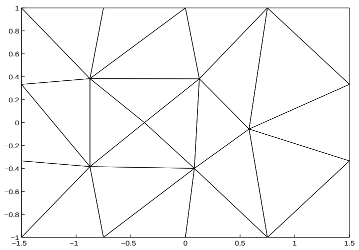


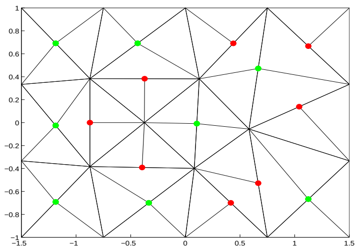


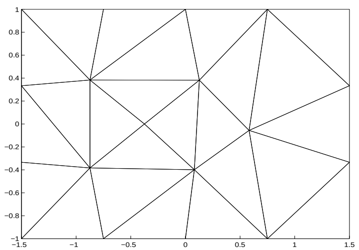


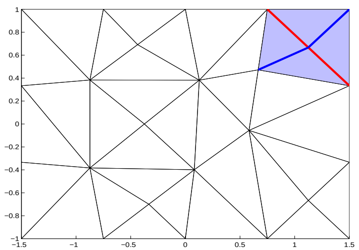


Hanging Nodes Do Not Appear

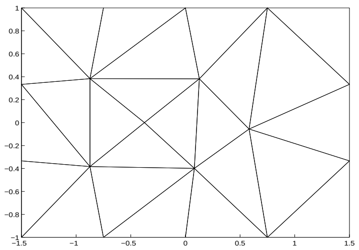


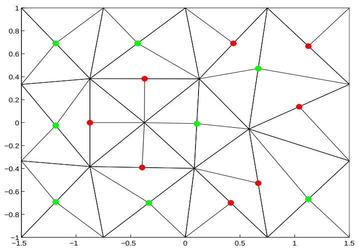


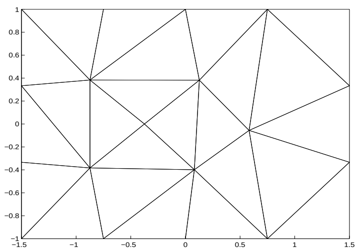


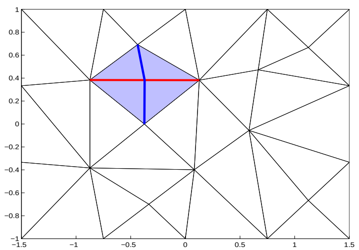


Hanging Nodes Do Not Appear

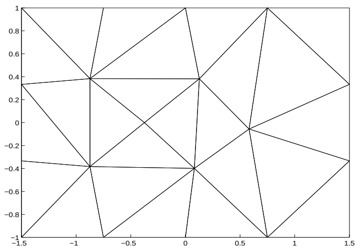


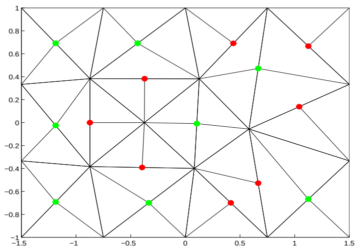


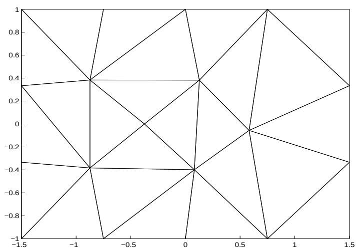


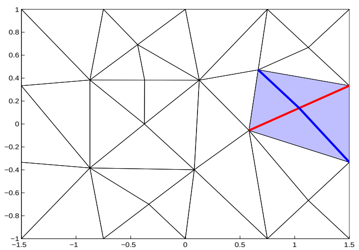


Hanging Nodes Do Not Appear

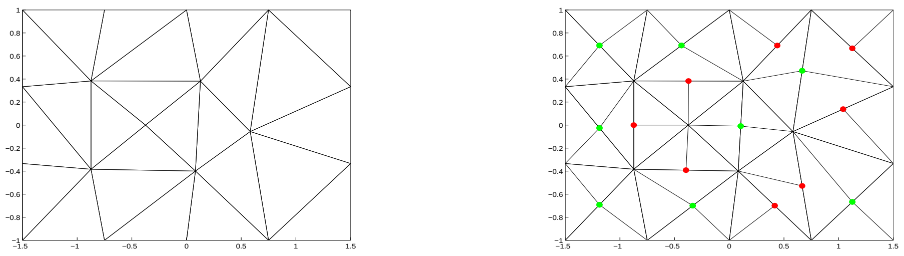


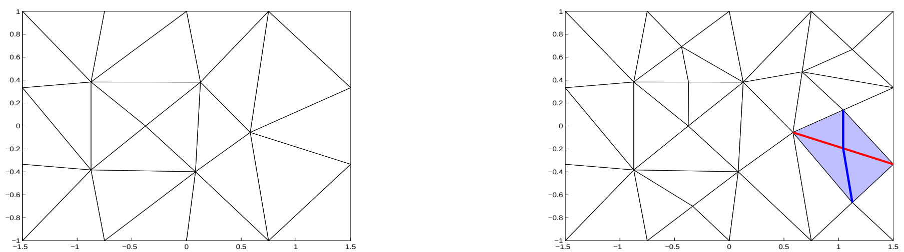




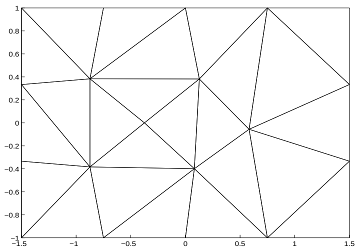


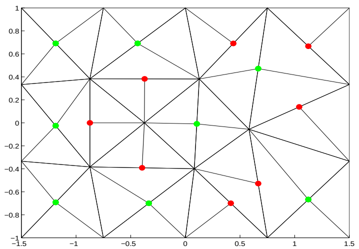
Hanging Nodes Do Not Appear

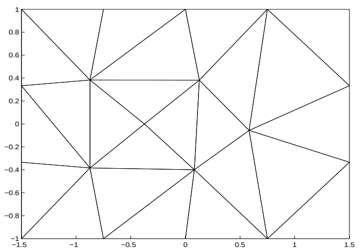


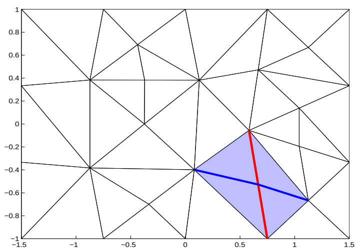


Hanging Nodes Do Not Appear

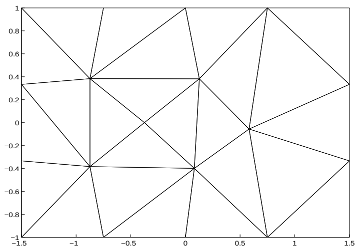


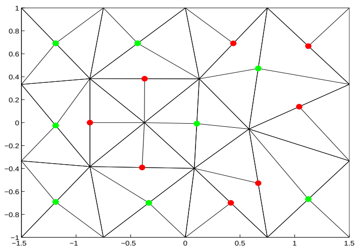


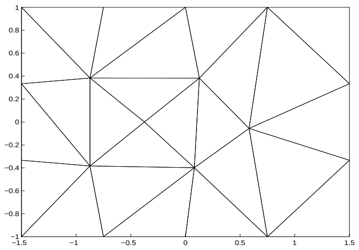


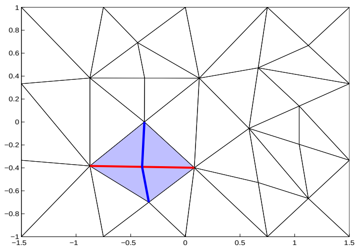


Hanging Nodes Do Not Appear

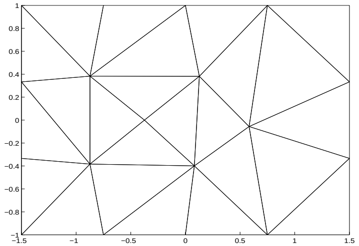


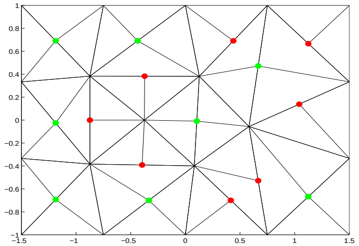


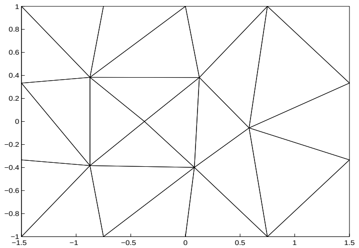


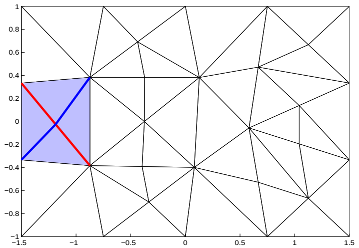


Hanging Nodes Do Not Appear

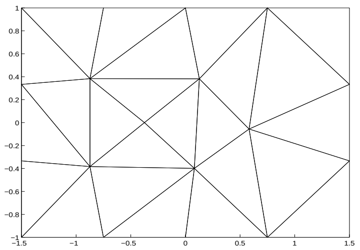


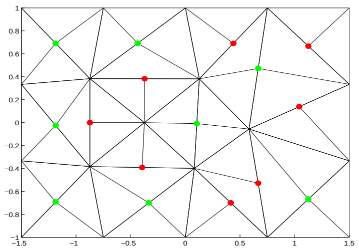


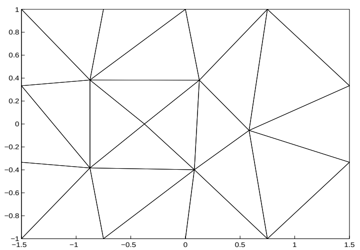


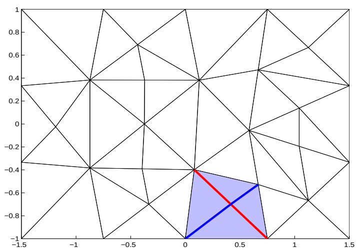


Hanging Nodes Do Not Appear

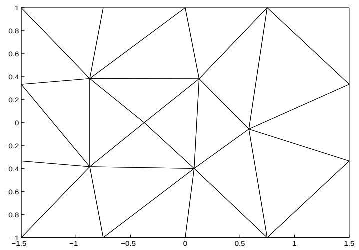


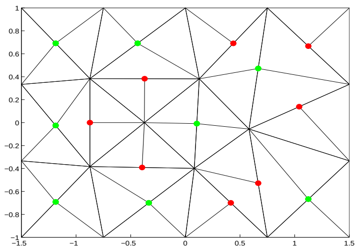


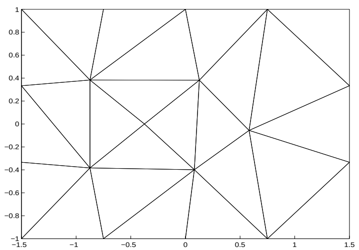


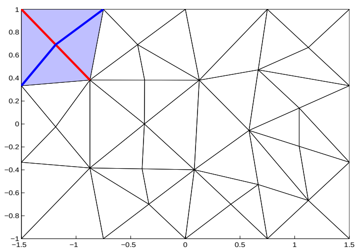


Hanging Nodes Do Not Appear

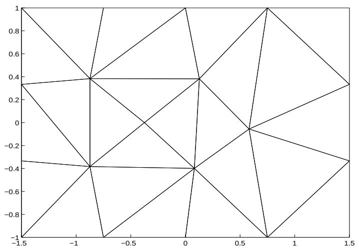


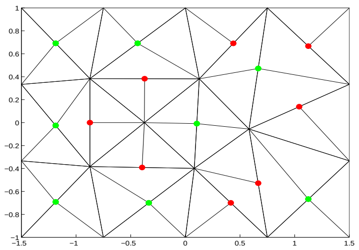


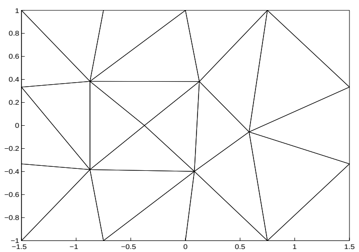


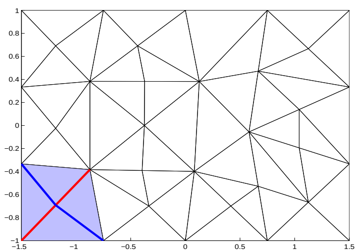


Hanging Nodes Do Not Appear

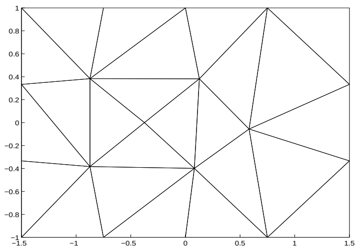


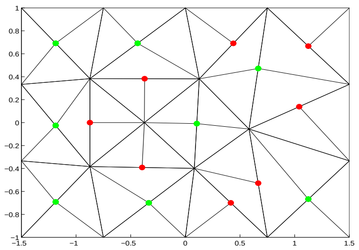


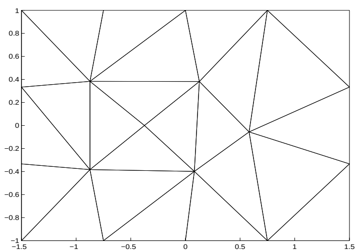


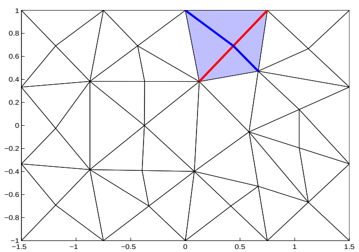


Hanging Nodes Do Not Appear

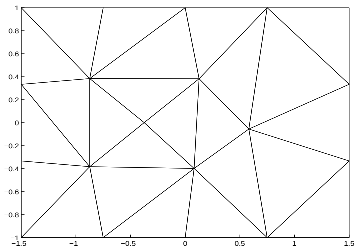


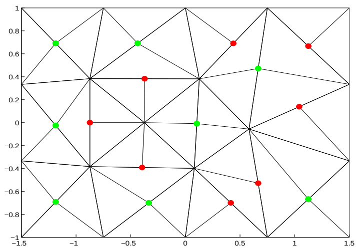


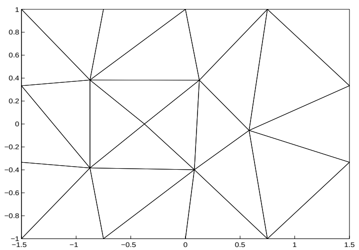


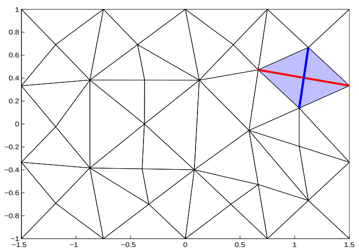


Hanging Nodes Do Not Appear

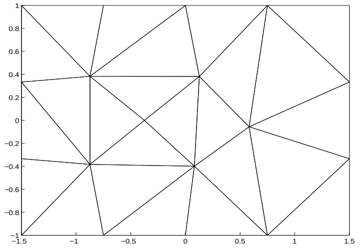


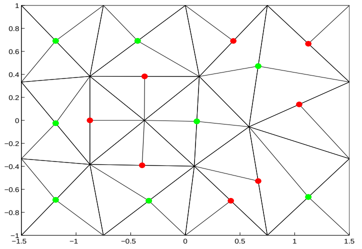


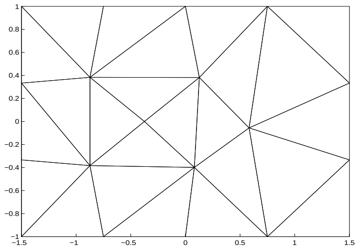


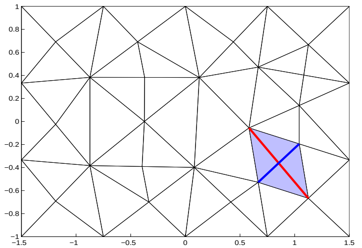


Hanging Nodes Do Not Appear

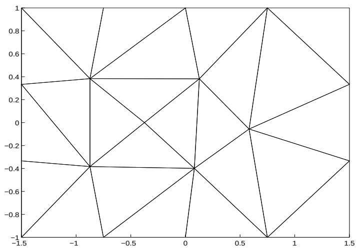


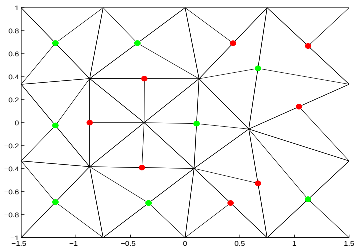


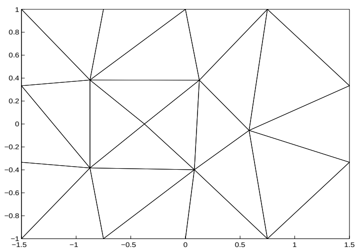


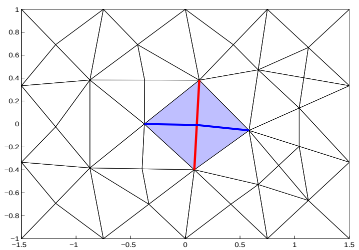


Hanging Nodes Do Not Appear

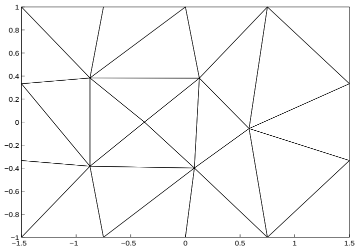


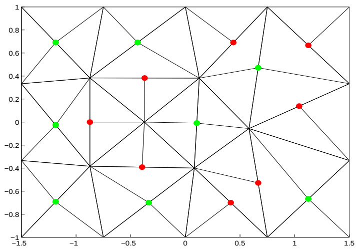


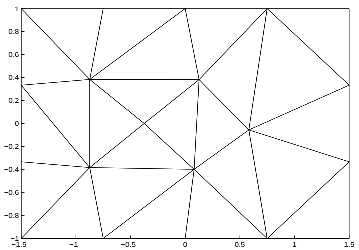


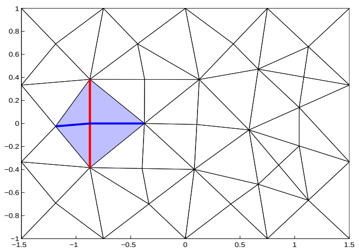


Hanging Nodes Do Not Appear

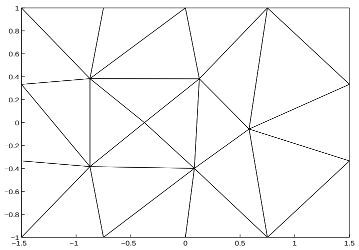


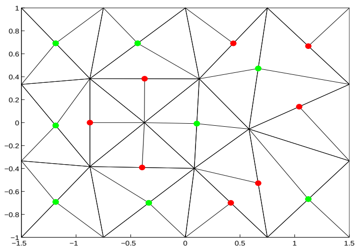


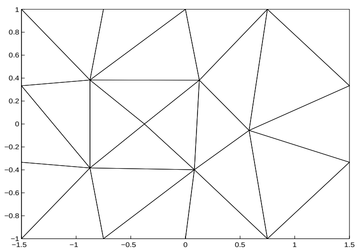


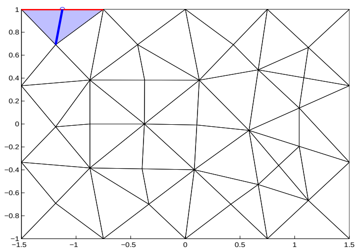


Hanging Nodes Do Not Appear

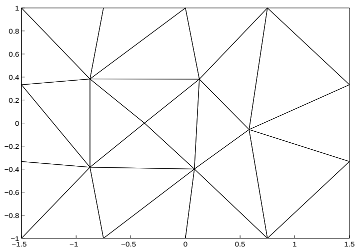


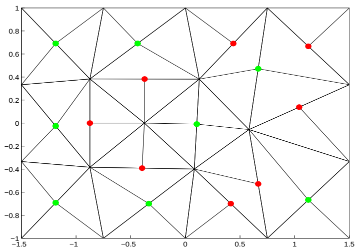


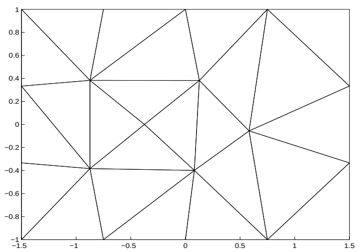


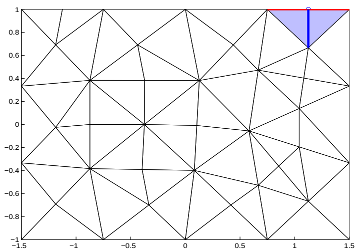


Hanging Nodes Do Not Appear

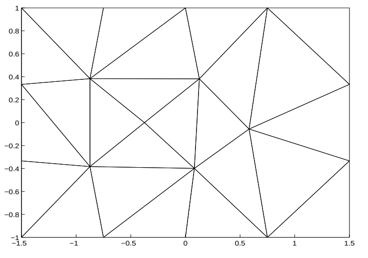


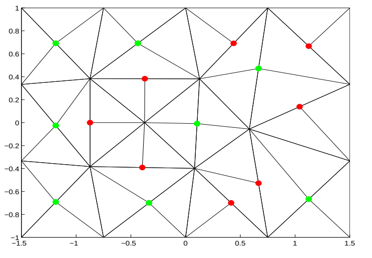


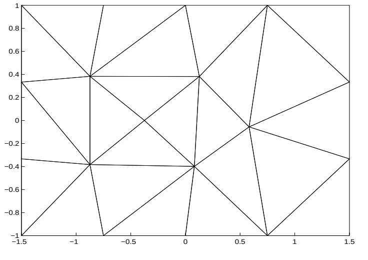


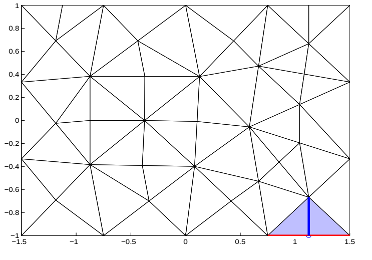


Hanging Nodes Do Not Appear

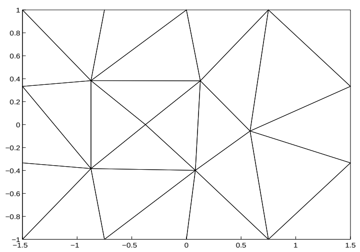


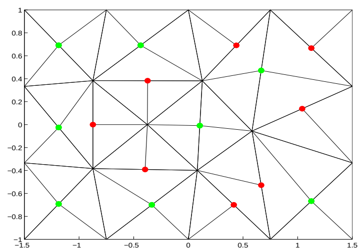


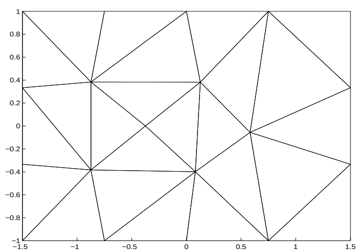


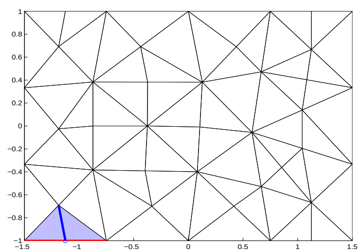


Hanging Nodes Do Not Appear

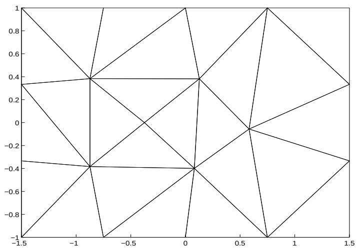


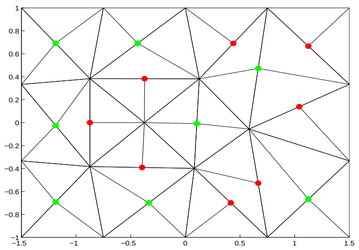


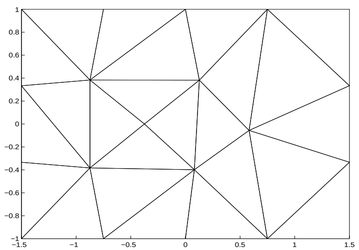


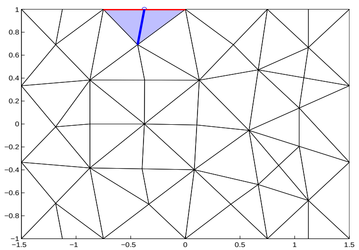


Hanging Nodes Do Not Appear

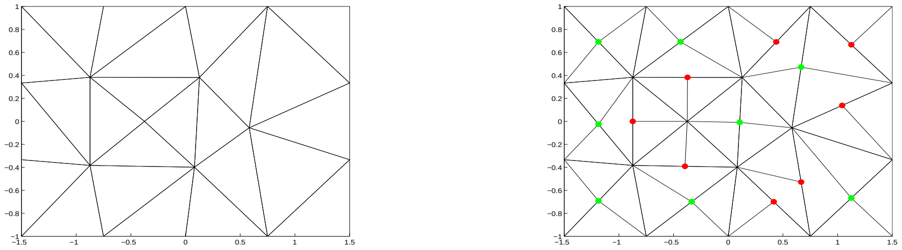


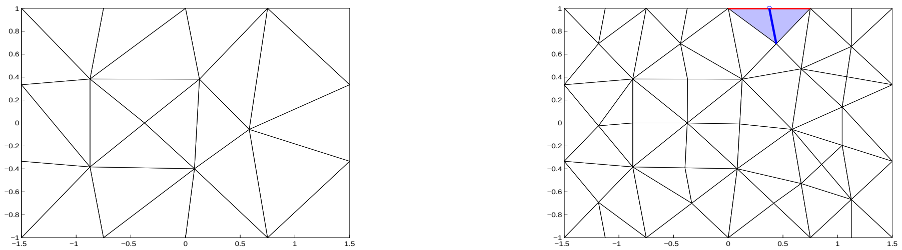




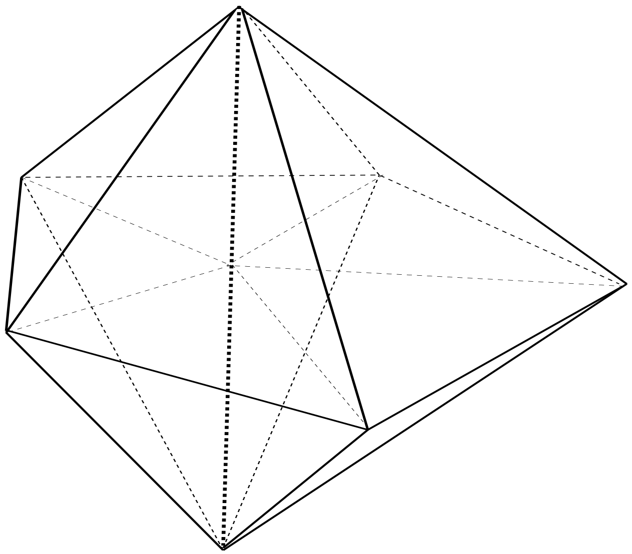


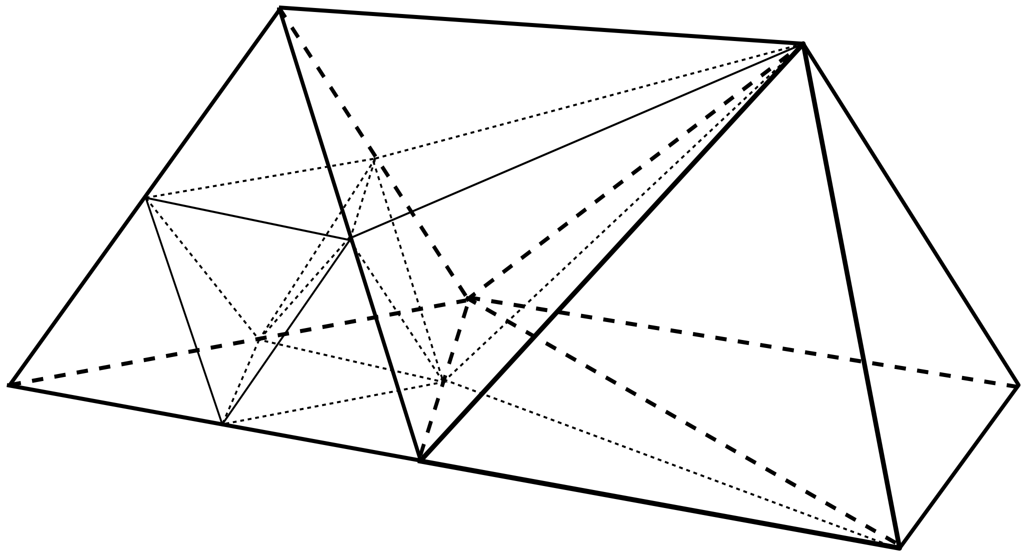
Hanging Nodes Do Not Appear





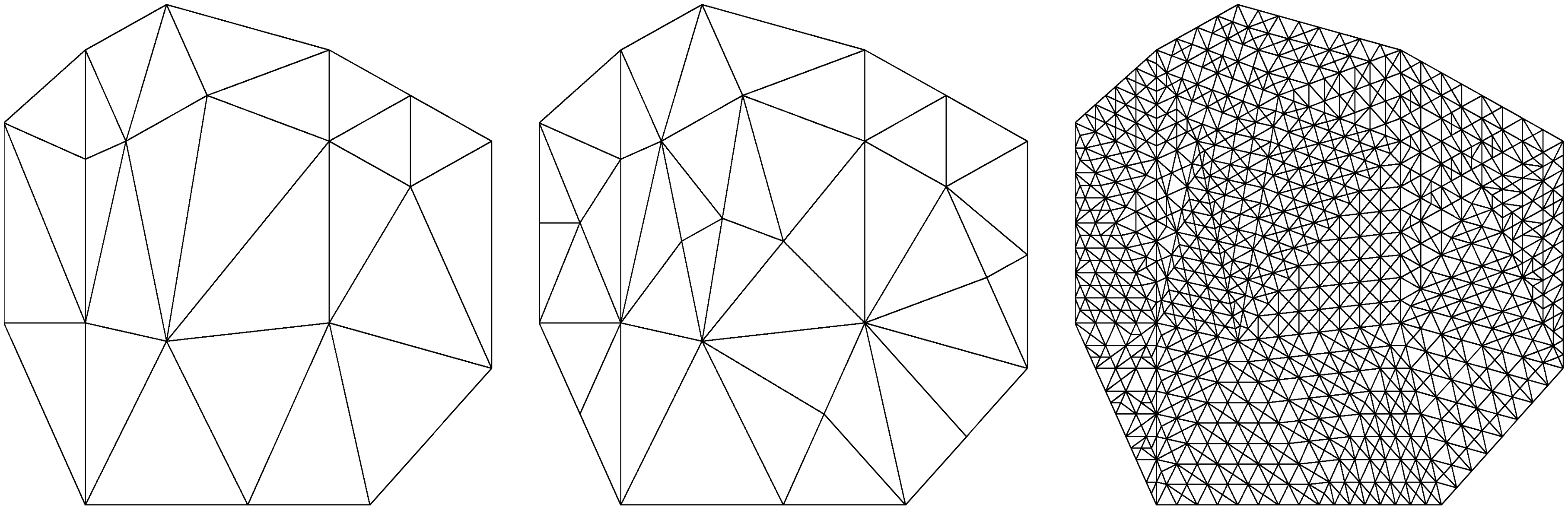
CLEB Refinement is Always Very Local



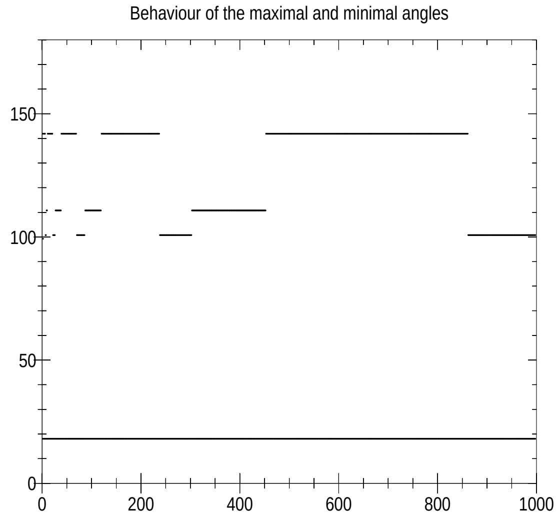


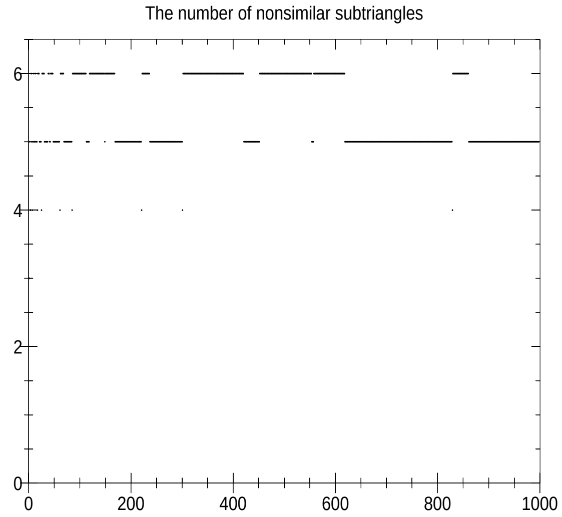
Numerical Experiments in 2d

• Performance of CLEB algorithm: initial triangulation, and triangulations after 10 and 1000 refinement steps



• We fix one triangle in the initial triangulation and monitor all refinements within it. What happens ?





- Angles in triangles are all between 18.5° and 143°
- Moreover, number of nonsimilar triangles is between 4 and 6

`>> The algorithm seems to produce a regular family of nested triangulations, since Zlámal's condition holds`

`>> Moreover, the subtriangles in triangulations are visually becoming of approximately the same size`

Regularity & Strong Regularity

- Zlámal's minimal angle condition is equivalent to the following

Definition: A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is called *regular* if there exists a constant $C > 0$ such that for all triangulations $\mathcal{T}_h \in \mathcal{F}$ and for all triangles $T \in \mathcal{T}_h$ we have

$$\operatorname{meas} T \geq C\, h_T^2$$

Definition: A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is called *strongly regular* if there exists a constant $C > 0$ such that for all triangulations $\mathcal{T}_h \in \mathcal{F}$ and for all triangles $T \in \mathcal{T}_h$ we have

$$\operatorname{meas} T \geq C\, h^2$$

Remark: Strong regularity implies regularity, but not vice versa.
Strong regularity $\equiv$ triangles are of approximately the same size

Math Results Proved for CLEB

Theorem 1: The CLEB algorithm yields a family of nested triangulations $\mathcal{F} = \{\mathcal{T}_h\}$, where h tends to zero monotonically

Theorem 2: Let α_0 be the minimal angle of all triangles from an initial triangulation. Then CLEB algorithm yields the following lower bound upon any angle α of any triangle from any $\mathcal{T}_h \in \mathcal{F}$

$$\alpha \geq \frac{\alpha_0}{2}$$

Theorem 3: The CLEB algorithm yields a strongly regular family of triangulations $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$

Modification of CLEB Suitable for Adaptivity

In the described above form, CLEB does not allow any local mesh refinement, it only solves the problem of generating a (strongly) regular family of (nested) triangulations (with some guaranteed properties)

• Our very recent efforts have been focused on a development of some modification of CLEB which could be used also for mesh adaptivity

A. Hannukainen, S. Korotov, M. Křížek. On global and local mesh refinements by a generalized conforming bisection algorithm. J. Comput. Appl. Math. ***235*** *(2010), 419–436*

Main Idea in Short

• The idea of a new variant of CLEB suitable for mesh adaptivity is to use some (positive) *mesh density function,* defined over the whole solution domain and coming e.g. from a posteriori error estimation, or defined in some another way a priori. Such function should be large over those parts of $\overline{\Omega}$ where we need a very fine mesh and small over those parts of $\overline{\Omega}$ where we do not need a fine mesh.

• `Then, with each edge in the mesh, we associate a number equal to the product of the lenth of this edge and the value of the mesh density function taken e.g. at the midpoint of this edge`

• Further, as for CLEB, we bisect the elements around that edge which has the largest number. It is clear that this procedure can be easily used in any dimension. *Defining a mesh density function we thus dictate the "adaptivity shape" for generating meshes. No hanging nodes, lines, or hyperlines appear !*

Current Theoretical Result on GCB

• The work on this algorithm, called *Generalized Conforming Bisection,* is currently in progress ...

Let the mesh density function m be Lipschitz continuous, i.e.

$$|m(x) - m(y)| \leq L\,|x - y|, \qquad x, y \in \overline{\Omega}$$

From the positiveness and continuity of m we have

$$0 < m_0 \leq m(x) \qquad \forall x \in \overline{\Omega}$$

Theorem: GCB algorithm yields a family of nested conforming triangulations $\mathcal{F} = \{\mathcal{T}_h\}_{h \to 0}$ if the initial mesh $\mathcal{T}_{init}$ satisfies the following condition

$$L_T \operatorname{diam} T \leq 0.03 \min_{x \in T} m(x) \qquad \forall T \in \mathcal{T}_{init},$$

where L_T is the minimal possible Lipschitz constant of m on T

GCB in L-Shaped Domain

Set the mesh density function as

$$m(x) = \frac{1}{1 + 4|x|} \tag{1}$$

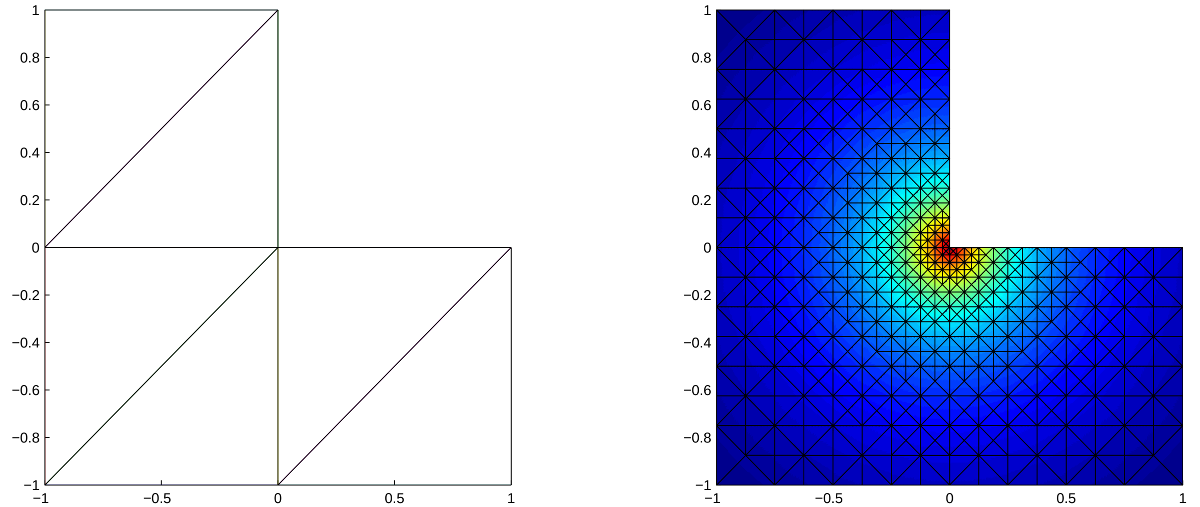


Figure 1: The initial mesh over solution domain (left). The right picture shows the behaviour of function (1).

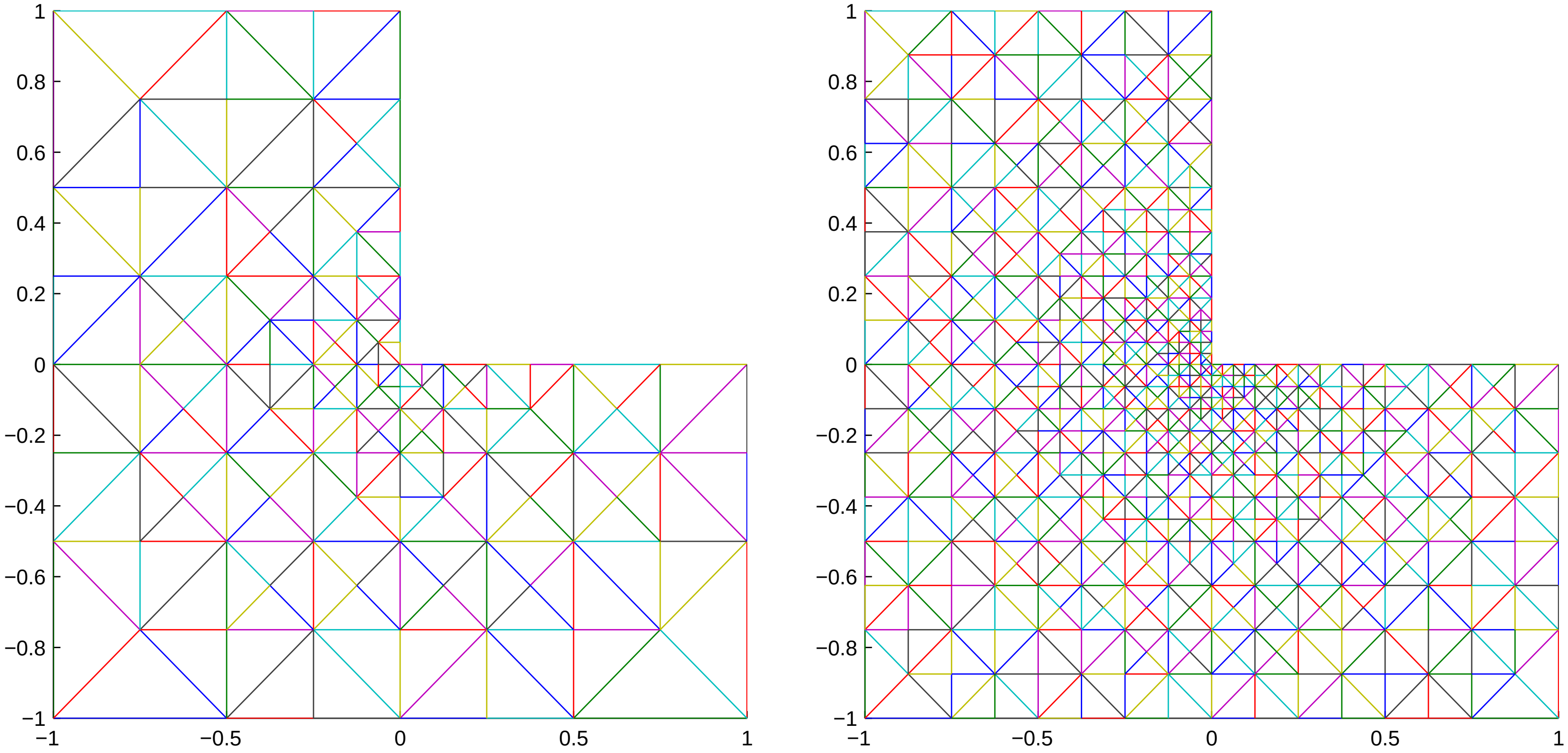


Figure 2: Resulting meshes after 100 and 500 refinements.

GCB for Boundary Layer

The domain in $\Omega = (-1,1)^2$ and $K = \left\{ (x,y) \,\middle|\, x = -1 \right\}$. The applied mesh density function for iterations $1 - 499$ is

$$m_1(x) = \frac{1}{0.1 + \text{dist}(K, x)} \tag{2}$$

and for iterations $500 - 1000$

$$m_2(x) = \frac{1}{0.01 + \text{dist}(K, x)}. \tag{3}$$

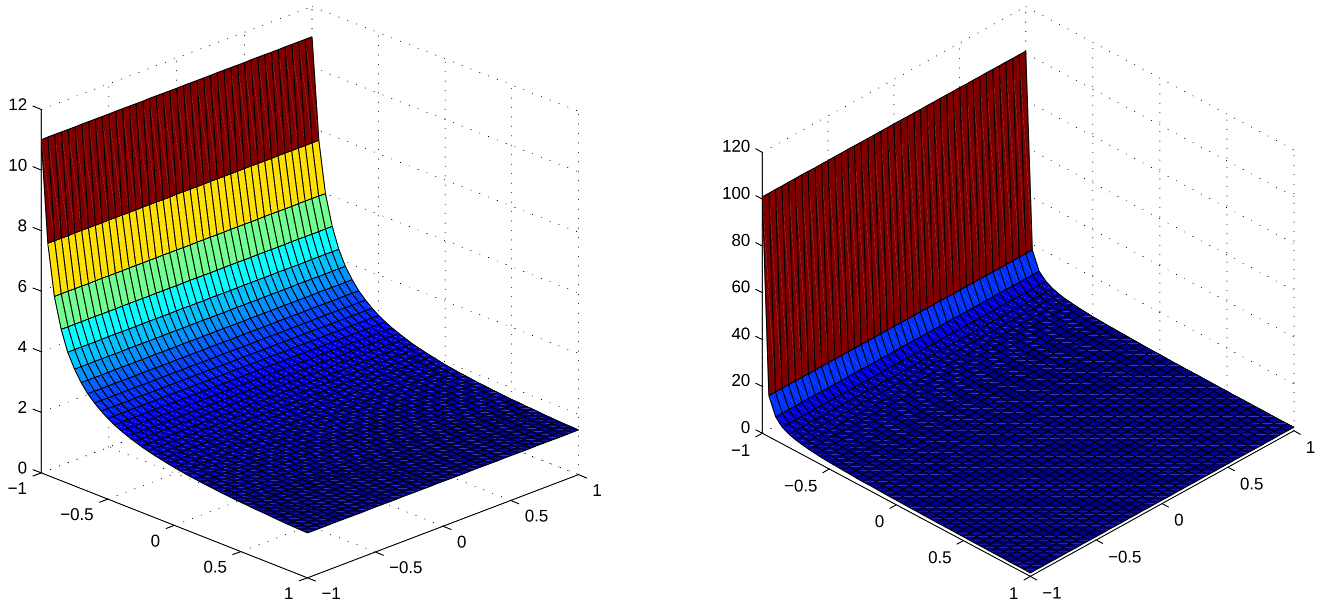


Figure 3: Mesh density function for iterations $1 - 499$ on left and mesh density function for iterations $500 - 1000$ on right.

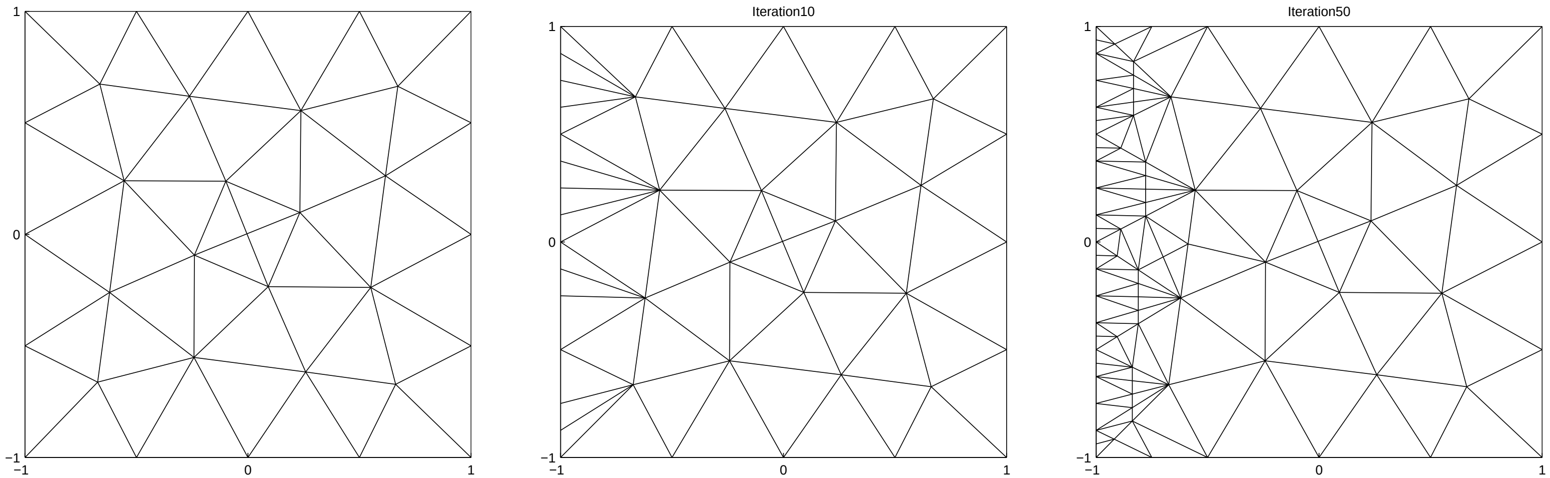


Figure 4: Initial mesh, meshes after 10 and 50 iterations.

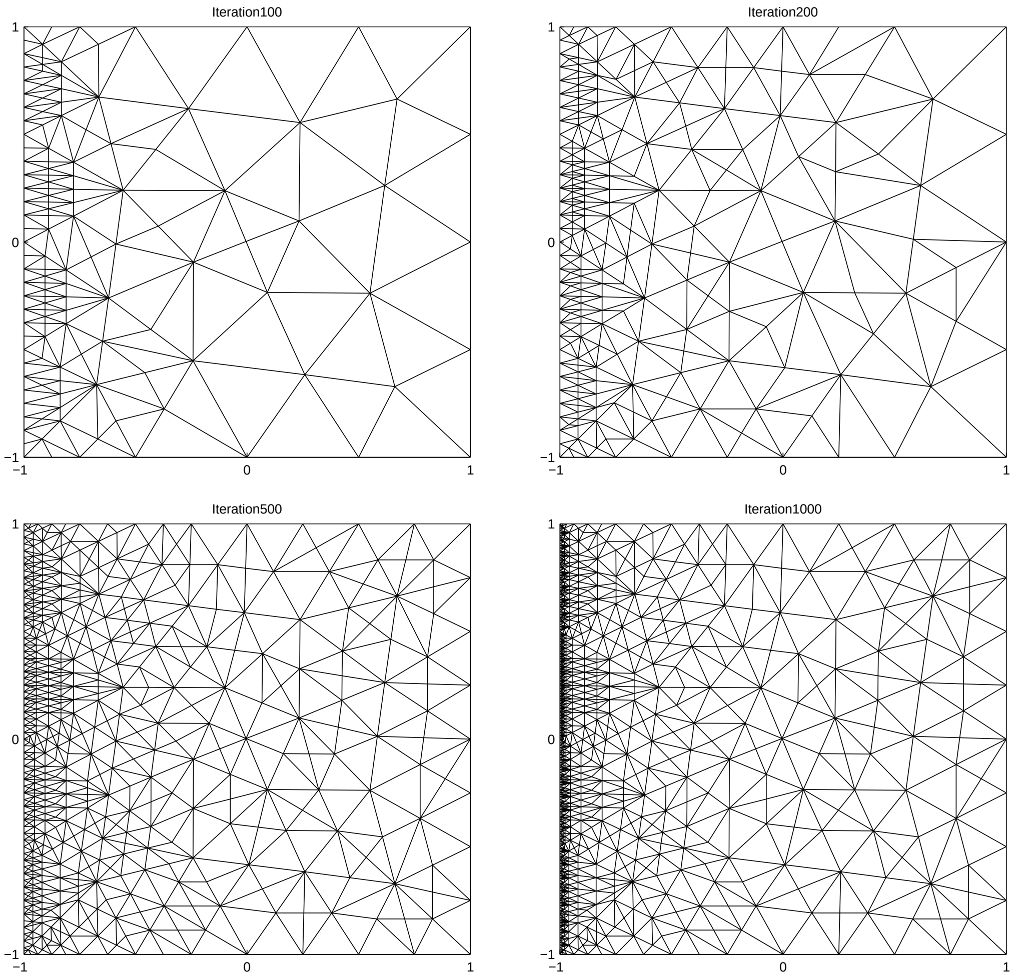


Figure 5: Meshes after 100, 200, 500, and 1000 iterations.

GCB for Interior Layer

The domain in $\Omega = (-1,1)^2$ and $K = \left\{ (x,y) \,\middle|\, y = x \right\}$. The applied mesh density function for iterations $1-499$ is

$$m_1(x) = \frac{1}{0.1 + \text{dist}(K,x)} \tag{4}$$

and for iterations $500-1000$

$$m_2(x) = \frac{1}{0.01 + \text{dist}(K,x)}. \tag{5}$$

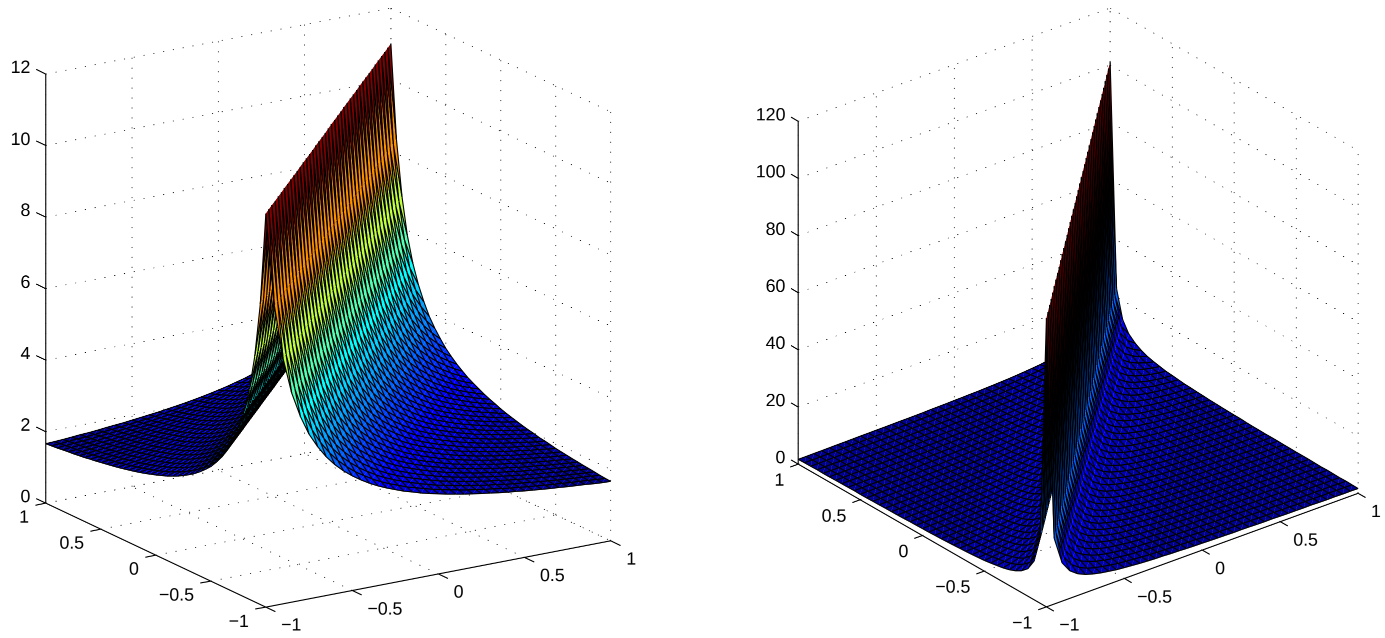


Figure 6: Mesh density function for iterations $1 - 499$ on left, and mesh density function for iterations $500 - 1000$ on right.

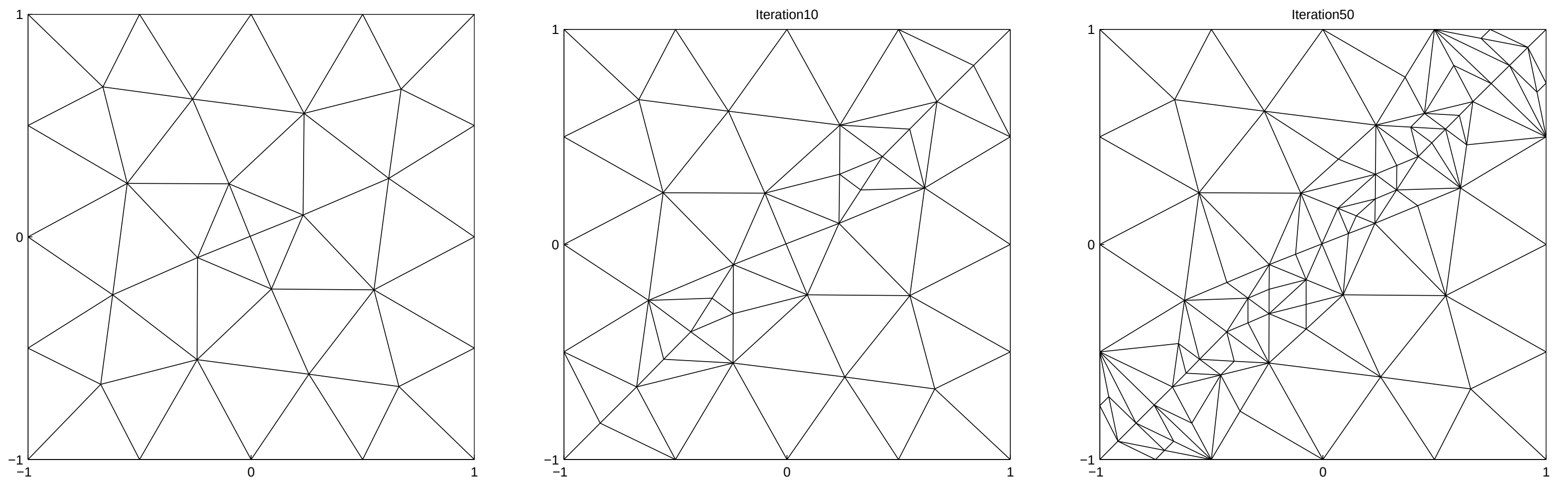


Figure 7: Initial mesh, meshes after 10 and 50 iterations.

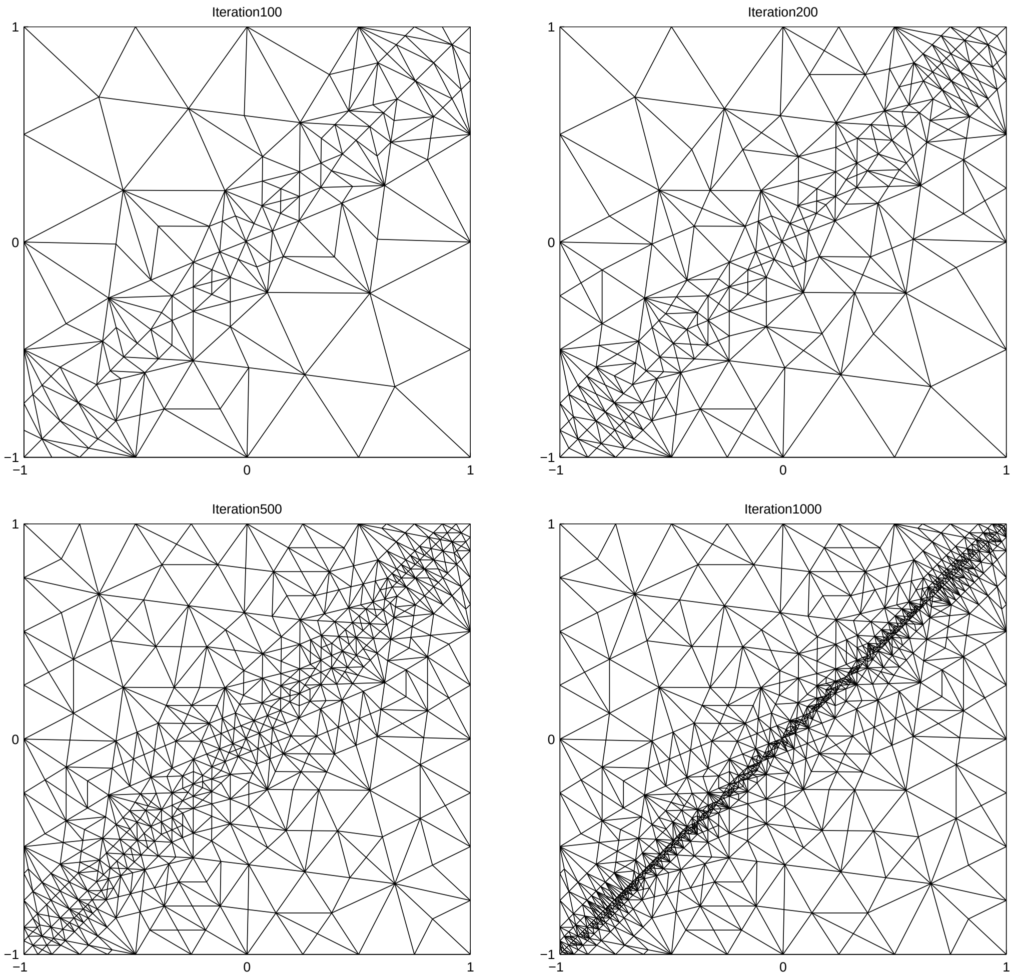
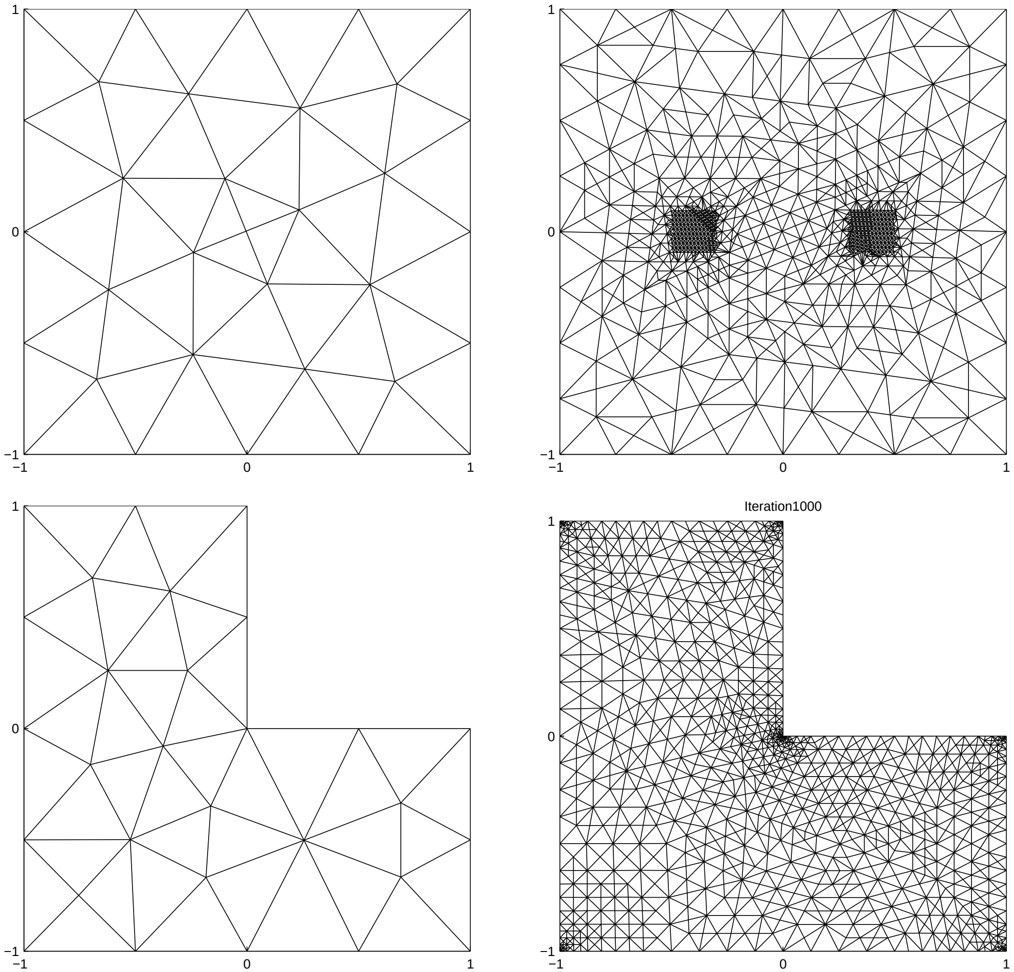


Figure 8: Resulting meshes after 100, 200, 500, and 1000 iterations.

More Examples



New General Result / Open Problem

Theorem: Let $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ be a family of simplicial partitions of a polyhedron $\overline{\Omega} \subset \mathbf{R}^d$ generated by CLEB algorithm. Then $\mathcal{F}$ is regular if and only if it is strongly regular

Remark: `The result is valid for any dimension`

Open problem: Prove some regularity results in higher dimensions, e.g. for $3d$ case

New Effects in $3d$

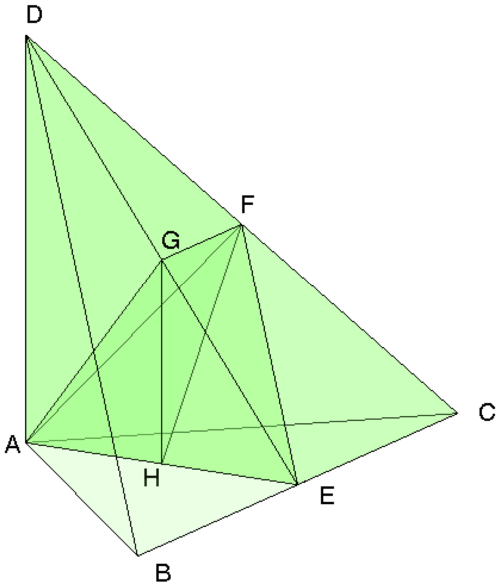
Example: Set $A = (-1, 0, 0)$, $B = (1, 0, 0)$, $C = (-2, 2, -1)$, and $D = (2, 2, 1)$. Then we have a tetrahedron $ABCD$ such that:

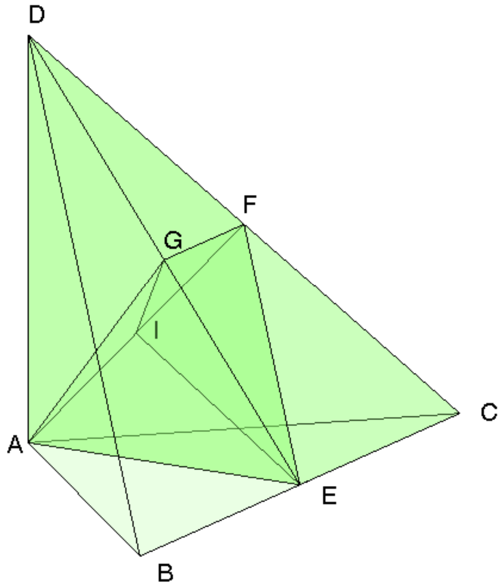
$$25.21^\circ \text{ at } |CD| = 4.47, \qquad 28.56^\circ \text{ at } |BD| = |AC| = 2.45,$$

$$53.13^\circ \text{ at } |AB| = 2, \qquad 133.09^\circ \text{ at } |BC| = |AD| = 3.75.$$

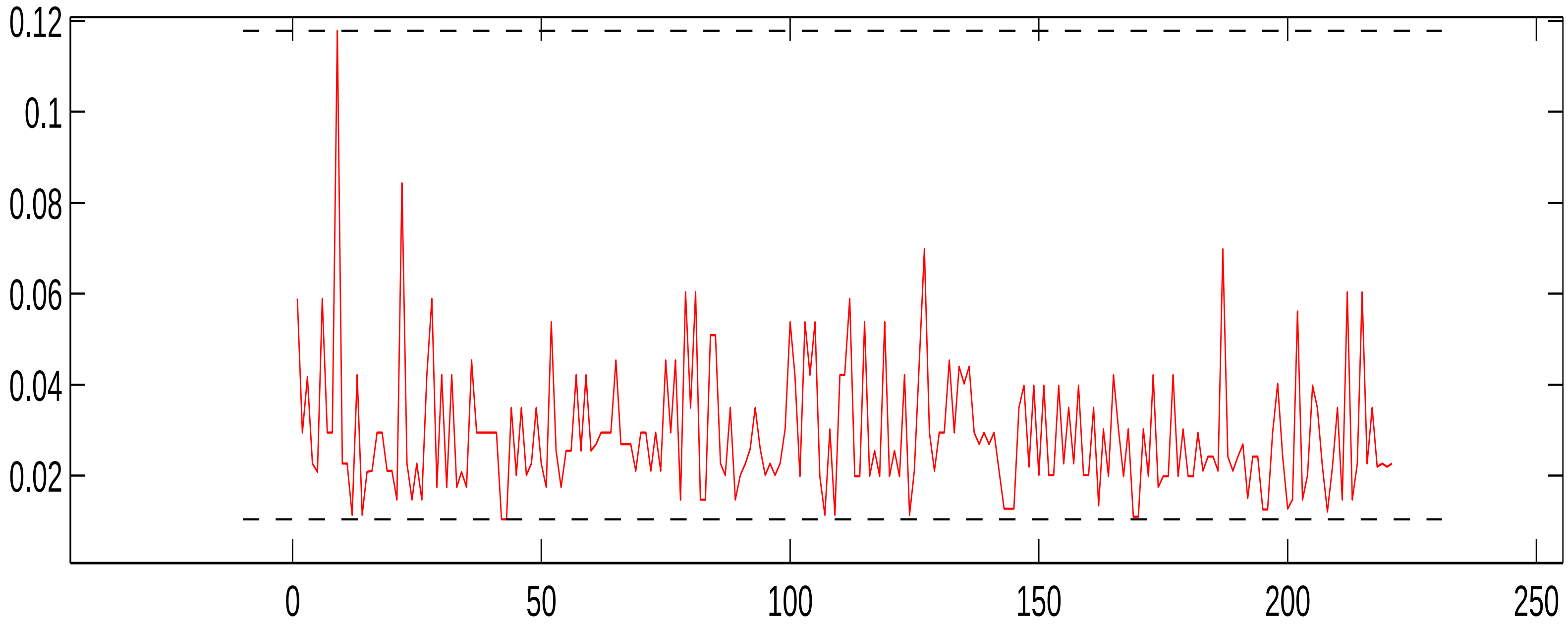
Observation: The largest (dihedral) angle is not opposite to the longest edge CD

CLEB does not produce uniquely determined shapes of subtetrahedra





The cube corner tetrahedron: the algorithm generates the following points $E = (B + C)/2$, $F = (C + D)/2$, and $G = (D + E)/2$. Then two edges AE and AF have the same length, but subtetrahedra $AFGH$ and $EFGI$ have different shapes, where $H = (A + E)/2$ and $I = (A + F)/2$, e.g. $AHGF$ is path, whereas $EFGI$ is not



Nevertheless, during the bisection process of the cube corner tetrahedron, the minimal aspect ratio $\mathrm{vol}\, S/(\mathrm{diam}\, S)^3$ seems to be numerically bounded from below by the positive constant $C = 0.0104$. The number of bisections is marked on the horizontal axis

`Similar result holds for many other shapes of tetrahedra, so we have a hope to get the regularity proved in 3`d

THANK YOU FOR YOUR ATTENTION !